

Continuity

1892 | The Law of Mind | W 8:143-145; CP 6.120-123

We now come to the difficult question, What is continuity? Kant confounds it with infinite divisibility, saying that the essential character of a continuous series is that between any two members of it a third can always be found. This is an analysis beautifully clear and definite; but, unfortunately, it breaks down under the first test. For according to this, the entire series of rational fractions arranged in the order of their magnitude would be an infinite series, although the rational fractions are numerable, while the points of a line are innumerable. Nay, worse yet, if from that series of fractions any two with all that lie between them be excised, and any number of such finite gaps be made, Kant's definition is still true of the series, though it has lost all appearance of continuity.

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Kant's definition expresses one simple property of a continuum; but it allows of gaps in the series. To mend the definition, it is only necessary to notice how these gaps can occur. Let us suppose, then, a linear series of points extending from a point, A, to a point, B, having a gap from B to a third point, C, and thence extending to a final limit, D; and let us suppose this series conforms to Kant's definition. Then, of the two points, B and C, one or both must be excluded from the series; for otherwise, by the definition, there would be points between them. That is, if the series contains C, though it contains all the points up to B, it cannot contain B. What is required, therefore, is to state in non-metrical terms that if a series of points up to a limit is included in a continuum the limit is included. It may be remarked that this is the property of a continuum to which Aristotle's attention seems to have been directed when he defines a continuum as something whose parts have a common limit. The property may be exactly stated as follows: If a linear series of points is continuous between two points, A and D, and if an endless series of points be taken, the first of them between A and D and each of the others between the last preceding one and D, then there is a point of the continuous series between all that endless series of points and D, and such that every other point of which this is true lies between this point and D. For example, take any number between 0 and 1, as 0.1; then, any number between 0.1 and 1, as 0.11; then any number between 0.11 and 1, as 0.111; and so on, without end. Then, because the series of real numbers between 0 and 1 is continuous, there must be a *least* real number, greater than every number of that endless series. This property, which may be called the Aristotelicity of the series, together with Kant's property, or its Kanticity, completes the definition of a continuous series.

The property of Aristotelicity may be roughly stated thus: a continuum contains the end point belonging to every endless series of points which it contains. An obvious corollary is that every continuum contains its limits. But in using this principle it is necessary to observe that a series may be continuous except in this, that it omits one or both of the limits.

1893 [c.] | Fallibilism, Continuity, and Evolution [R] | CP 1.171

... let me call your attention to the natural affinity of this principle to the doctrine of fallibilism. The principle of continuity is the idea of fallibilism objectified. For fallibilism is the doctrine that our

knowledge is never absolute but always swims, as it were, in a continuum of uncertainty and of indeterminacy. Now the doctrine of continuity is that *all things* so swim in continua.

1893 [c.] | Fallibilism, Continuity, and Evolution [R] | CP 1.163-164

But in order really to see all there is in the doctrine of fallibilism, it is necessary to introduce the idea of continuity, or unbrokenness. This is the leading idea of the differential calculus and of all the useful branches of mathematics; it plays a great part in all scientific thought, and the greater the more scientific that thought is; and it is the master key which adepts tell us unlocks the arcana of philosophy.

We all have some idea of continuity. Continuity is fluidity, the merging of part into part. But to achieve a really distinct and adequate conception of it is a difficult task, which with all the aids possible must for the most acute and most logically trained intellect require days of severe thought.

1896 [c.] | Lessons of the History of Science | CP 1.62

It is not necessary to read far in almost any work of philosophy written by a man whose training is that of a theologian, in order to see how helpless such minds are in attempting to deal with continuity. Now continuity, it is not too much to say, is the leading conception of science. The complexity of the conception of continuity is so great as to render it important wherever it occurs. Now it enters into every fundamental and exact law of physics or of psychics that is known. The few laws of chemistry which do not involve continuity seem for the most part to be very roughly true. It seems not unlikely that if the veritable laws were known continuity would be found to be involved in them...

1898 | Training in Reasoning | RLT 190

Generality [...] is logically the same as continuity. But continuity is Thirdness in its full entelechy.

1902 | Synechism | CP 6.172

True generality is [...] nothing but a rudimentary form of true continuity. Continuity is nothing but perfect generality of a law of relationship.

1903 | Peirce's Personal Interleaved Copy of the 'Century Dictionary' [Commens] | CP 6.166

...I made a new definition, according to which continuity consists in *Kanticity* and *Aristotelicity*. The *Kanticity* is having a point between any two points. The *Aristotelicity* is having every point that is a limit to an infinite series of points that belong to the system.

I here slightly modify Cantor's definition of a perfect system. Namely, he defines it as such that it contains every point in the neighborhood of an infinity of points and no other. But the latter is a character of a concatenated system; hence I omit it as a character of a perfect system.

But further study of the subject has proved that this definition is wrong. It involves a misunderstanding of Kant's definition which *he himself* likewise fell into. Namely he defines a continuum as that all of whose parts have parts of the same kind. He himself, and I after him, understood that to mean infinite divisibility, which plainly is not what constitutes continuity since the series of rational fractional values is infinitely divisible but is not by anybody regarded as continuous. Kant's real definition implies that a continuous line contains no points. Now if we are to accept the common sense idea of continuity (after correcting its vagueness and fixing it to mean something) we must either say that a continuous line contains no points or we must say that the principle of excluded middle does not hold of these points. The principle of excluded middle only applies to an individual (for it is not true that "Any man is wise" nor that "Any man is not wise"). But places, being mere possibles without actual existence, are not individuals. Hence a point or indivisible place really does not exist unless there actually be something there to mark it, which, if there is, interrupts the continuity. I, therefore, think that Kant's definition correctly defines the common sense idea, although there are great difficulties with it. I certainly think that on any line whatever, on the common sense idea, there is room for any multitude of points however great. If so, the *analytical continuity* of the theory of functions, which implies there is but a single point for each distance from the origin, defined by a quantity expressible to indefinitely close approximation by a decimal carried out to an indefinitely great number of places, is certainly not the continuity of common sense, since the whole multitude of such quantities is only the first abnumeral multitude, and there is an infinite series of higher grades. On the whole, therefore, I think we must say that continuity is the relation of the parts of an unbroken space or time. The precise definition is still in doubt; but Kant's definition, that a continuum is that of which every part has itself parts of the same kind, seems to be correct. This must not be confounded (as Kant himself confounded it) with infinite divisibility, but implies that a line, for example, contains no points until the continuity is broken by marking the points. In accordance with this it seems necessary to say that a continuum, where it is continuous and unbroken, contains no definite parts; that its parts are created in the act of defining them and the precise definition of them breaks the continuity. In the calculus and theory of functions it is assumed that between any two rational points (or points at distances along the line expressed by rational fractions) there are rational points and that further for every convergent series of such fractions (such as 3.1, 3.14, 3.141, 3.1415, 3.14159, etc.) there is just one limiting point; and such a collection of points is called *continuous*. But this does not seem to be the common sense idea of continuity. It is only a collection of independent points. Breaking grains of sand more and more will only make the sand more broken. It will not weld the grains into unbroken continuity.

1908-05-24 | Supplement | CP 7.535 n. 6

A perfect continuum belongs to the genus, of a whole all whose parts without any exception whatsoever conform to one general law to which same law conform likewise all the parts of each single part. *Continuity* is thus a special kind of *generality*, or conformity to one Idea. More specifically, it is a *homogeneity*, or generality among all of a certain kind of parts of one whole. Still more specifically, the characters which are the same in all the parts are a certain kind of relationship of each part to all the coordinate parts; that is, it is a *regularity*. The step of specification which seems called for next, as appropriate to our purpose of defining, or logically analyzing the Idea of continuity, is that of asking ourselves what kind [of] relationship between parts it is that constitutes the regularity a continuity; and

the first, and therefore doubtless the best answer for our purpose, not as the ultimate answer, but as the proximate one, is that it is the relation or relations of *contiguity*; for continuity is unbrokenness (whatever that may be,) and this seems to imply a *passage* from one part to a contiguous part. What is this 'passage'? This passage seems to be an act of turning the attention from one part to another part; in short an actual event in the mind. This seems decidedly unfortunate, since an event can only take place in Time, and Time is a continuum; so that the prospect is that we shall rise from our analysis with a definition of continuity in general in terms of a special continuity. However, it is possible that this objection will disappear as we proceed.